(Oxford U. P., Oxford, England, 1961), Sec. 4.9; Ref. 2, Fig. 2. Atom *B* represents the metal atom and the positive direction of the reference axis is directed from *B* to *A*.

[9] Here we do not include explicitly the effective field correction; part of this correction is contained implicitly in the macroscopic parameters in the final expression for $\chi^{(2)}$. To allow for it completely is difficult in the present simple model.

[10] See Ref. 2; $\beta_{\parallel} \gg \beta_{\perp}$ reflects the fact that the electronic distribution of the atoms forming the bond changes mainly along the bond axis while the charge distribution transversally to the bond axis tends to retain that of the atom with spherical symmetry. For an isolated atom $\beta \equiv 0$.

[11] J. Ducuing and Chr. Flytzanis, in *Optical Properties of Solids*, edited by F. Abeles (North-Holland, Amsterdam, 1970), Sec. 3.3 and the appendices; for CuCl see Sec. 3.3.

[12] I. Freund and B. F. Levine, Phys. Rev. Letters 25, 1241 (1970). In this paper an x-ray diffraction technique is proposed to measure the modulation of the bonding charge distribution by the optical field. Similar techniques should give instead some information about the effective field corrections associated with the optical field. See also R. Martin and J. A. van Vechten (private communication).

[13] J. C. Phillips, Phys. Rev. 166, 832 (1968); see also J. Walter and M. L. Cohen, Phys. Rev. Letters 26, 17 (1970).

[14] J. Cižek, Mol. Phys. 6, 19 (1963).

[15] R. C. Miller and W. Nordland, Opt. Commun. 1, 400 (1970).

[16] B. N. Figgis, *Introduction to Ligand Fields* (Wiley, New York, 1966).

[17] It is known that the electronegativity difference alone does not predict uniquely the polarity of the bond. See also L. Pauling, *The Nature of the Chemical Bond* (Cornell U. P., Ithaca, New York, 1960) for a discussion of the electroneutrality principle in bonds.

[18] One of us (C. F.) acknowledges discussion with Dr. J. A. van Vechten on this point.

[19] In all these compounds (ZnO, CuCl, InN, and GaN) the electronegativity difference ΔX (Pauling) is larger than unity or $\Delta X > 1$; for the zinc-blende compounds considered in Table I on the other hand $\Delta X < 1$. This separation in two classes with $\chi^{(2)} < 0$ if $\Delta X > 1$ and $\chi^{(2)} > 0$ if $\Delta X < 1$, however, seems to be more general (compare BeO).

[20] K. S. Song, J. Phys. (Paris) 28, 195 (1967).

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Analytical Solutions of the Boltzmann Transport Equation. I. Carrier Transport Phenomena in Nondegenerate Semiconductors at Low Fields

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By expressing the carrier distribution function in a power series, a self-consistent method is presented for solving analytically the Boltzmann transport equation. Using an iterative technique, this method enables the calculation of carrier mobilities in nondegenerate semiconductors to any desired degree of accuracy at fields of any strengths. A "mesh" diagram is proposed for the calculation of the low-field mobilities governed by polar-optical scattering. It is shown that the results obtained by relaxation-time approximation under some conditions are rather poor. The effect of applied magnetic field is also discussed in some detail.

I. INTRODUCTION

The theory of carrier transport phenomena in semiconductors lies on the determination of the carrier distribution function f which is governed by the applied field $\vec{\xi}$, and the scattering of the carriers with the carriers and the lattice vibrations. In the steady state these effects are balanced in accordance with the Boltzmann equation

$$e\vec{\xi}\cdot\frac{\partial f(\vec{p})}{\partial \vec{p}} + \sum_{\vec{p}} W(\vec{p}\to\vec{p}\,')f(\vec{p}) - \sum_{\vec{p}\,'} W(\vec{p}\,'\to\vec{p})f(\vec{p}\,') + L_{ee}f(\vec{p}) = 0 , \qquad (1)$$

where e is the electronic charge, $W(\vec{p}\to\vec{p}\,')$ is the transition probability per unit time of a carrier in a state with momentum $\vec{p}$ being scattered into a state with momentum $\vec{p}\,'$ by the lattice vibrations, and L_{ee} is the probability per unit time of a carrier being scattered by another carrier. Thus each term in this equation represents a rate of change of f; the term $e\vec{\xi}\cdot\partial f(\vec{p})/\partial\vec{p}$ is due to the applied field $\vec{\xi}$, the terms $\sum_{\vec{p}\,'} W(\vec{p}\to\vec{p}\,')f(\vec{p})$ and $\sum_{\vec{p}\,'} W(\vec{p}\,'\to\vec{p})f(\vec{p}\,')$ are, respectively, due to the carriers being scattered out of and into state $\vec{p}$ through the absorption or emission of phonons, and the term $L_{ee}f(\vec{p})$ is due to the carrier-carrier scattering. Since there is no net carrier transport in the absence of external fields, the equilibrium distribution function can be assumed to be the Maxwell-Boltzmann function for nondegenerate semiconductors when $\vec{\xi} = 0$.

In the low-field transport theory the effect of carrier-carrier scattering is generally ignored because the carrier concentration in nondegenerate semiconductors is so small that the carriers are mostly scattered by the lattice vibrations.[1] The effect of the electron-phonon scattering is usually treated by the relaxation-time approximation. With this method the off-equilibrium part of the distribution function is assumed to be small and to relax with an energy-dependent time constant which can be calculated from the transition probabilities $W(\vec{p}\rightarrow\vec{p}\,')$ and $W(\vec{p}\,'\rightarrow\vec{p})$. Actually, the relaxation-time approximation is valid only for the scattering which is elastic or nearly elastic and isotropic, and therefore it is not applicable to the polar semiconductors in which the dominant scattering is highly inelastic and anisotropic. To avoid using the relaxation-time concept, Howarth and Sondheimer,[2] and Ehrenreich[3] used a variational approach to calculate the low-field carrier mobility in polar semiconductors, and recently Rode[4] used a simple iterative technique to solve the same problem. However, both the variational and iterative methods can lead to an exact solution only in the limit of vanishingly small fields. The accuracy of these methods for calculating the low-field carrier mobility at fields of finite strengths can be assessed only by comparing them with a method which can give an exact solution for fields of any strengths.

The purpose of this paper is to present a self-consistent method of solving analytically the Boltzmann equation and calculating the carrier mobility. This method is similar to the iterative method[4] and has all the advantages of it. Matthiessen's rule is totally avoided, and the exact solution of the linearized Boltzmann equation expressed in the form of a monotonic series lends itself to straightforward physical interpretation. Moreover, this method has its own merits; it is valid for any types of band structure and can be easily extended to study all transport phenomena at a field of any strengths, and the rate of convergence of the monotonic series can be easily determined.

In this paper we present the calculations of carrier mobilities and related transport phenomena in nondegenerate semiconductors at low fields. The calculations of carrier mobilities at high fields will be presented in the next paper in this series.

II. ANALYTICAL SOLUTION OF BOLTZMANN EQUATION

When electron-electron scattering is ignored and no external magnetic field is present, the Boltzmann equation becomes

$$eF\,\frac{\partial f}{\partial p_F}+\frac{f}{\tau}-Hf=0\ , \tag{2}$$

in which, p_F, τ, and H are defined as

$$p_F=\vec{p}\cdot\vec{F}/F\ , \tag{3}$$

$$1/\tau=\textstyle\sum_{\vec{p}\,'}W(\vec{p}\rightarrow\vec{p}\,')\ , \tag{4}$$

$$H\phi=\textstyle\sum_{\vec{p}\,'}W(\vec{p}\,'\rightarrow\vec{p})\,\phi(\vec{p}\,')\ , \tag{5}$$

where $\vec{F}$ is the applied electric field and ϕ is an arbitrary function of $\vec{p}$.

To separate the low- and high-field transport phenomena, we expand f into a power series in F,

$$f=f_0+Ff_1+F^2f_2+\cdots+F^nf_n+\cdots\ , \tag{6}$$

where f_0 is the Maxwell-Boltzmann function $A\,e^{-E/k_BT}$, A is the normalization constant, E is the electron energy, k_B is the Boltzmann constant, and T is the lattice temperature. Equation (6) is the solution of Eq. (2) provided that the series converges. The functions $f_1, f_2, \ldots, f_n, \ldots$ can be determined by the following set of coupled equations which are obtained by substituting Eq. (6) into Eq. (2) and equating coefficients of individual powers of F:

$$\begin{aligned}&e\,\frac{\partial f_0}{\partial p_F}+\frac{f_1}{\tau}-Hf_1=0\ ,\\ &e\,\frac{\partial f_1}{\partial p_F}+\frac{f_2}{\tau}-Hf_2=0\ ,\ \ldots\\ &e\,\frac{\partial f_n}{\partial p_F}+\frac{f_{n+1}}{\tau}-Hf_{n+1}=0\ ,\ \ldots\ .\end{aligned} \tag{7}$$

The first equation in Eq. (7) is the linearized Boltzmann equation which accounts for all the low-field transport phenomena. It is obvious that f_{n+1} can be determined from the knowledge of f_n, and that if a method can be devised to solve the first equation exactly for f_1, the same method can be applied to find the exact solutions for all f_n's for $n\geqslant 2$, since all the equations governing f_n for $n\geqslant 1$ are of the same form. This method of expressing the distribution function to successive orders in the applied electric field is similar to the one proposed by Schottky.[5] But unlike Schottky who expressed the f_n's as solutions of a Fredholm type of integro-differential equation, we solve for the f_n's explicitly by a self-consistent method as described below.

By neglecting the "scattering-in" term Hf_1 as a first approximation, we obtain

$$f_1=-\,e\tau\,\frac{\partial f_0}{\partial p_F}\ . \tag{8}$$

Using this form of f_1 for Hf_1 in Eq. (7), the second approximation for f_1 is

$$f_1=-\,e\tau(1+H\tau)\,\frac{\partial f_0}{\partial p_F}\ . \tag{9}$$

The continued application of this iterative technique yields the exact solution for f_1 as

$$f_1=e\tau(1+H\tau+H\tau H\tau+\cdots)\,\frac{\partial f_0}{\partial p_F}\ . \tag{10}$$

The exactness of Eq. (10) can easily be verified by substituting it into Eq. (7).

The physical meaning of Eq. (10) can be seen by writing $H\tau\, \partial f_0/\partial p_F$ in terms of the transition probabilities

$$H\tau \frac{\partial f_0}{\partial p_F} = \sum_{\vec{p}'} \left(W(\vec{p}' \to \vec{p}) \Big/ \sum_{\vec{p}''} W(\vec{p}' \to \vec{p}'') \right) \left[\frac{\partial f_0}{\partial p_F} \right]_{\vec{p} = \vec{p}'} . \tag{11}$$

The term $W(\vec{p}' \to \vec{p}) / \sum_{\vec{p}''} W(\vec{p}' \to \vec{p}'')$ is the ratio of the number of electrons in state $\vec{p}'$ being scattered into state $\vec{p}$ to the total number of electrons being scattered out of state $\vec{p}'$. The transition of electrons from one state to another is accompanied with absorption and emission of phonons. By changing the summation over $\vec{p}'$ and $\vec{p}''$ to the summation over $\vec{q}$, the phonon momentum, Eq. (11) becomes

$$H\tau \frac{\partial f_0}{\partial p_F} = \sum_{\vec{q}} \left(W(\vec{p}+\vec{q} \to \vec{p}) \Big/ \sum_{\vec{q}'} [W(\vec{p}+\vec{q} \to \vec{p}+\vec{q}+\vec{q}') + W(\vec{p}+\vec{q} \to \vec{p}+\vec{q}-\vec{q}')] \right) \left[\frac{\partial f_0}{\partial p_F} \right]_{\vec{p} = \vec{p}+\vec{q}}$$
$$+ \sum_{\vec{q}} \left(W(\vec{p}-\vec{q} \to \vec{p}) \Big/ \sum_{\vec{q}'} [W(\vec{p}-\vec{q} \to \vec{p}-\vec{q}+\vec{q}') + W(\vec{p}-\vec{q} \to \vec{p}-\vec{q}-\vec{q}')] \right) \left[\frac{\partial f_0}{\partial p_F} \right]_{\vec{p} = \vec{p}-\vec{q}} . \tag{12}$$

It is now obvious that $H\tau\, \partial f_0/\partial p_F$ is the contribution to f_1 due to the transition of electrons from the neighboring states $\vec{p}+\vec{q}$ and $\vec{p}-\vec{q}$ to state $\vec{p}$ by emission and absorption of one phonon. Similarly, $H\tau H\tau\, \partial f_0/\partial p_F$ can be written as

$$H\tau H\tau \frac{\partial f_0}{\partial p_F} = \sum_{\vec{q}} \left(W(\vec{p}+\vec{q} \to \vec{p}) \Big/ \sum_{\vec{q}'} [W(\vec{p}+\vec{q} \to \vec{p}+\vec{q}+\vec{q}') + W(\vec{p}+\vec{q} \to \vec{p}+\vec{q}-\vec{q}')] \right) \left[H\tau \frac{\partial f_0}{\partial p_F} \right]_{\vec{p} = \vec{p}+\vec{q}}$$
$$+ \sum_{\vec{q}} \left(W(\vec{p}-\vec{q} \to \vec{p}) \Big/ \sum_{\vec{q}'} [W(\vec{p}-\vec{q} \to \vec{p}-\vec{q}+\vec{q}') + W(\vec{p}-\vec{q} \to \vec{p}-\vec{q}-\vec{q}')] \right) \left[H\tau \frac{\partial f_0}{\partial p_F} \right]_{\vec{p} = \vec{p}-\vec{q}} . \tag{13}$$

This is the sum of four terms. Each term describes the effect of two successive transitions, and each transition is accompanied with either a phonon-emission or a phonon-absorption process. For example, one of the terms on the right-hand side of Eq. (13) is

$$\sum_{\vec{q}} \left(W(\vec{p}+\vec{q} \to \vec{p}) \Big/ \sum_{\vec{q}'} [W(\vec{p}+\vec{q} \to \vec{p}+\vec{q}+\vec{q}') + W(\vec{p}+\vec{q} \to \vec{p}+\vec{q}-\vec{q}')] \right) \sum_{\vec{q}''} \left(W(\vec{p}+\vec{q}+\vec{q}'' \to \vec{p}+\vec{q}) \Big/ \right.$$
$$\left. \sum_{\vec{q}'''} [W(\vec{p}+\vec{q}+\vec{q}'' \to \vec{p}+\vec{q}+\vec{q}''+\vec{q}''') + W(\vec{p}+\vec{q}+\vec{q}'' \to \vec{p}+\vec{q}+\vec{q}''-\vec{q}''')] \right) \left[\frac{\partial f_0}{\partial p_F} \right]_{\vec{p} = \vec{p}+\vec{q}+\vec{q}''} , \tag{14}$$

which corresponds to the two successive transitions by phonon emission $\vec{p}+\vec{q}+\vec{q}'' \to \vec{p}+\vec{q} \to \vec{p}$. In general, $H\tau H\tau \cdots H\tau$ n times operating on $\partial f_0/\partial p_F$ is the sum of 2^n terms each of which corresponds to n successive transitions, and each transition involves either a phonon-emission or phonon-absorption process.

Using the same iterative technique which leads to Eq. (10), we can obtain

$$f_2 = \left(-e\tau(1 + H\tau + H\tau H\tau + \cdots) \frac{\partial}{\partial p_F} \right)^2 f_0 \tag{15}$$

and, in general,

$$f_n = \left(-e\tau(1 + H\tau + H\tau H\tau + \cdots) \frac{\partial}{\partial p_F} \right)^n f_0 . \tag{16}$$

Thus the distribution function f can be expressed by

$$f = f_0 + \sum_{n=1}^{\infty} \left(-eF\tau(1 + H\tau + H\tau H\tau + \cdots) \frac{\partial}{\partial p_F} \right)^n f_0 . \tag{17}$$

For a given band structure, Eq. (17) can be used to calculate the mobility due to any type of electron-phonon scattering at any field strengths. The case when more than one type of electron-phonon scattering are operative presents no formal problem, and this will be discussed briefly in Sec. III.

Since $1/\tau$ is proportional to the coupling constant F_l which characterizes the coupling strength between the electrons and the lattice and has the dimension of field strength, the rate of convergence of the series in Eq. (17) is determined, among other factors (such as temperature), by the ratio F/F_l. Thus it is safe to state that the present low-field mobility calculation by keeping only the terms linear in F in Eq. (17) is accurate up to the order of F/F_l. In the following we present our low-field transport calculations. The discussion of high-field transport phenomena will be presented in the next paper in this series.

III. LOW-FIELD ELECTRON MOBILITY

For fields small enough such that $F^2 f_2 \ll F f_1$ the distribution function can be written as

$$f = f_0 + F f_1 \tag{18}$$

and the electron mobility as

$$\mu_L = \int f_1 \frac{\partial E}{\partial p_F} \, d\vec{p} \Big/ \int f_0 \, d\vec{p} \ , \tag{19}$$

where E is the electron energy. The calculation now lies on the determination of f_1 which is governed by the types of scattering. Three cases are discussed.

A. Relaxation-Time Approximation

If the scattering processes are assumed to be isotropic or nearly elastic and isotropic, our method will give the results identical to those obtained by the relaxation-time approximation.[6] For isotropic scattering processes, the transition probability $W(\vec{p} \to \vec{p}\,')$ is an even function of the electron velocity $\partial E/\partial p_F$. Since $\partial f_0/\partial p_F$ is equal to $-(f_0/k_B T)\partial E/\partial p_F$, the term $H\tau \, \partial f_0/\partial p_F$ vanishes and f_1 is then given by

$$f_1 = -e\tau \frac{\partial f_0}{\partial p_F} \ . \tag{20}$$

For elastic scattering processes, $f_0(\vec{p}) = f_0(\vec{p}\,')$ and if in addition $W(\vec{p} \to \vec{p}\,')$ depends only on the relative orientation of $\vec{p}$ and $\vec{p}\,'$ and is independent of the direction of $\vec{p}$ or $\vec{p}\,'$, then $\tau(\vec{p}) = \tau(\vec{p}\,')$ and $W(\vec{p} \to \vec{p}\,') = W(\vec{p}\,' \to \vec{p})$. We can obtain

$$H\tau \frac{\partial f_0}{\partial p_F} = -\tau \frac{f_0}{k_B T} \frac{\partial E}{\partial p_F} \sum_{\vec{p}\,'} [W(\vec{p} \to \vec{p}\,') \cos\phi] \ , \tag{21}$$

where ϕ is the angle between $\vec{p}$ and $\vec{p}\,'$. From Eq. (9) we can express f_1 as

$$f_1 = e\tau \frac{f_0}{k_B T} \frac{\partial E}{\partial p_F} g(E) \ . \tag{22}$$

The function $g(E)$ is determined by rewriting Eq. (10) as

$$(1 - H\tau) f_1 = e\tau \frac{f_0}{k_B T} \frac{\partial E}{\partial p_F} \tag{23}$$

and substituting Eq. (22) into Eq. (23). Thus, we have

$$\tau g(E) \{1 - \tau \textstyle\sum_{\vec{p}\,'} [W(\vec{p} \to \vec{p}\,') \cos\phi]\} = \tau \ . \tag{24}$$

From Eq. (4), f_1 is then given by

$$f_1 = \left(e \Big/ \sum_{\vec{p}\,'} [W(\vec{p} \to \vec{p}\,')(1 - \cos\phi)] \right) \frac{f_0}{k_B T} \frac{\partial E}{\partial p_F} \ . \tag{25}$$

Theoretically the electron-phonon scattering is always inelastic to some degree since the energies of an electron before and after scattering differ by the energy of a phonon. The usual argument is that if the energy of the phonon is very much smaller than the energy of the electron the scattering can be considered to be elastic. For the validity of Eq. (25) the scattering must at the same time be not extremely anisotropic as pointed out by Herring and Vogt.[7] Extreme anisotropy implies that many terms on the right-hand side of Eq. (10) have to be included to give a fair approximation to f_1. It has been shown in Sec. II that $H\tau H\tau \cdots H\tau$ n times operating on $\partial f_0/\partial p_F$ consists of 2^n terms each of which corresponds to n successive transitions, and each transition involves a phonon-emission or phonon-absorption process. The importance of these terms is that when n is large enough to make $n \times$ (phonon energy) to become comparable to the electron energy, the scattering cannot be considered as elastic. However, our method provides a means to test quantitatively the validity of Eq. (25). If appreciable convergence can be obtained only after the inclusion of n terms in Eq. (10), where $n \times$ (phonon energy) is comparable to the electron energy, the accuracy of using Eq. (25) for low-field mobility calculations may be rather poor.

B. Polar-Optical Scattering

Although the acoustic-mode scattering may be considered as elastic, the polar-optical-mode scattering has been known to be highly anisotropic and inelastic. For the latter, the series for f_1 in Eq. (10) cannot be summed and expressed in a closed form. Nevertheless, it is possible to write down the general term of this series and then to examine its convergence without going through the laborious work in summing up the leading terms until the desired convergence is attained.

The transition probability for the polar-optical scattering, apart from a δ function expressing the conservation of energy, is given by[6]

$$W(\vec{p} \to \vec{p}\,') = \frac{2\pi\hbar^2 e F_0}{m q^2} \left(N_q + \frac{1}{2} + \frac{\delta N_q}{2} \right) \ , \tag{26}$$

$$e F_0 = \frac{m e^2 \hbar \, \omega_0}{\hbar^2} \left(\frac{1}{\mathcal{K}_\infty} - \frac{1}{\mathcal{K}_0} \right) \ . \tag{27}$$

m is the electron effective mass, $\hbar$ is the Planck constant divided by 2π, ω_0 is the angular frequency of polar-optical phonons, $\mathcal{K}_\infty, \mathcal{K}_0$ are the dielectric constants at infinite and zero frequencies, respectively, N_q is the number of polar-optical phonons and equals $[\exp(\hbar\,\omega_0/k_B T) - 1]^{-1}$, $\delta N_q = 1$ for phonon emission (by electron), and $\delta N_q = -1$ for phonon absorption (by electron).

Equation (26) is valid only for electrons at the wave vector $\vec{k}=0$ in the conduction band with wave function of s symmetry. As $\vec{k}$ departs from zero, there is an increasing admixture of the wave functions of p symmetry. This admixture is particularly important in semiconductors with small energy-band gaps. However, it is not expected to alter the anisotropic nature of the polar-optical scattering. On the contrary, the effect of screening, which is also ignored in Eq. (26), tends to reduce the scattering probability in the forward direction.[3] To take into account this effect, we introduce the Debye shielding length λ_D and rewrite the transition probability as

$$W(\vec{p}\to\vec{p}\,') = \frac{2\pi\hbar^2 eF_0}{mq^2}\,\frac{1}{(1+\hbar^2/q^2\lambda_D^2)^2}\left(N_q+\frac{1}{2}+\frac{\delta N_q}{2}\right), \qquad (28)$$

in which $\hbar$ appears in the screening factor because $\vec{q}$ denotes the phonon momentum. By assuming the band structure of the semiconductors to be parabolic for simplicty, Eq. (10) becomes

$$f_1 = \frac{e\tau}{mk_BT}\,(1+H\tau+H\tau H\tau+\cdots)\,p_F f_0 \,. \qquad (29)$$

To obtain the general term $H\tau H\tau\cdots H\tau$ n times operating on $p_F f_0$, we proceed as follows:

(i) From Eq. (4) and (29) it can be easily shown that

$$\begin{aligned}\frac{1}{\tau} = \frac{2eF_0}{(2mE)^{1/2}}\Bigg[&\frac{N_q}{4}\ln\frac{1+(E/E_s)[(1+\hbar\omega_0/E)^{1/2}+1]^2}{1+(E/E_s)[(1+\hbar\omega_0/E)^{1/2}-1]^2}+\frac{(N_q+1)}{4}\ln\frac{1+(E/E_s)[1+(1-\hbar\omega_0/E)^{1/2}]^2}{1+(E/E_s)[1-(1-\hbar\omega_0/E)^{1/2}]^2}\\ &+\frac{N_q}{4}\left(\frac{1}{1+(E/E_s)[(1+\hbar\omega_0/E)^{1/2}+1]^2}-\frac{1}{1+(E/E_s)[(1+\hbar\omega_0/E)^{1/2}-1]^2}\right)\\ &+\frac{N_q+1}{4}\left(\frac{1}{1+(E/E_s)[1+(1-\hbar\omega_0/E)^{1/2}]^2}-\frac{1}{1+(E/E_s)[1-(1-\hbar\omega_0/E)^{1/2}]^2}\right)\Bigg]\\ = \frac{2eF_0}{(2mE)^{1/2}}\,&\alpha\!\left(\frac{E}{\hbar\omega_0}\right), \end{aligned} \qquad (30)$$

where $E_s=\hbar^2/2m\lambda_D^2$, a quantity with the dimension of energy; the function $\alpha(E/\hbar\omega_0)$ is self-defined by Eq. (30).

(ii) There are two terms in $H\tau p_F f_0$. For the phonon-emission term, $\tau(E)$ and $f_0(E)$ can be taken out of the summation over $\vec{q}$ and replaced with $\tau(E+\hbar\omega_0)$ and $f_0(E+\hbar\omega_0)$. By performing the summation over $\vec{q}$, we obtain the phonon emission $(\vec{p}+\vec{q}\to\vec{p})$ term in $H\tau p_F f_0$ as

$$\begin{aligned}&\frac{N_q}{2}\left(1+\frac{\hbar\omega_0}{E}\right)^{1/2}\left[\left(2+\frac{\hbar\omega_0}{E}+2\frac{E_s}{E}\right)\frac{1}{4}\ln\frac{1+(E/E_s)[(1+\hbar\omega_0/E)^{1/2}+1]^2}{1+(E/E_s)[(1+\hbar\omega_0/E)^{1/2}-1]^2}+\left(2+\frac{\hbar\omega_0}{E}+\frac{E_s}{E}\right)\right.\\ &\quad\times\frac{1}{4}\left.\left(\frac{1}{1+(E/E_s)[(1+\hbar\omega_0/E)^{1/2}+1]^2}-\frac{1}{1+(E/E_s)[(1+\hbar\omega_0/E)^{1/2}-1]^2}\right)-\left(1+\frac{\hbar\omega_0}{E}\right)^{1/2}\right]\frac{1}{\alpha(E/\hbar\omega_0+1)}\,p_F f_0\\ &= \frac{N_q}{2}\,\beta_-\!\left(\frac{E}{\hbar\omega_0}\right)\frac{1}{\alpha(E/\hbar\omega_0+1)}\,p_F f_0 \;, \end{aligned} \qquad (31)$$

where the function $\beta_-(E/\hbar\omega_0)$ is self-defined by Eq. (31). Similarly, the phonon-absorption $(\vec{p}-\vec{q}\to\vec{p})$ term in $H\tau p_F f_0$ is

$$\begin{aligned}&\frac{(N_q+1)}{2}\left(1-\frac{\hbar\omega_0}{E}\right)^{1/2}\left[\left(2-\frac{\hbar\omega_0}{E}+2\frac{E_s}{E}\right)\frac{1}{4}\ln\frac{1+(E/E_s)[1+(1-\hbar\omega_0/E)^{1/2}]^2}{1+(E/E_s)[1-(1-\hbar\omega_0/E)^{1/2}]^2}+\left(2-\frac{\hbar\omega_0}{E}+\frac{E_s}{E}\right)\right.\\ &\quad\times\frac{1}{4}\left.\left(\frac{1}{1+(E/E_s)[1+(1-\hbar\omega_0/E)^{1/2}]^2}-\frac{1}{1+(E/E_s)[1-(1-\hbar\omega_0/E)^{1/2}]^2}\right)-\left(1-\frac{\hbar\omega_0}{E}\right)^{1/2}\right]\frac{1}{\alpha(E/\hbar\omega_0-1)}\,p_F f_0\\ &= \frac{(N_q+1)}{2}\,\beta_+\!\left(\frac{E}{\hbar\omega_0}\right)\frac{1}{\alpha(E/\hbar\omega_0-1)}\,p_F f_0 \;, \end{aligned} \qquad (32)$$

TABLE I. Four terms in $H\tau H\tau p_F f_0$.

Transition	Mathematical expression
$\vec{p}+2\vec{q}\rightarrow\vec{p}+\vec{q}\rightarrow\vec{p}$	$\frac{N_q}{2}\frac{\beta_-(E/\hbar\omega_0+1)}{\alpha(E/\hbar\omega_0+2)}\frac{N_q}{2}$
	$\times\frac{\beta_-(E/\hbar\omega_0)}{\alpha(E/\hbar\omega_0+1)}\,p_F f_0$
$\vec{p}-2\vec{q}\rightarrow\vec{p}-\vec{q}\rightarrow\vec{p}$	$\frac{(N_q+1)}{2}\frac{\beta_+(E/\hbar\omega_0-1)}{\alpha(E/\hbar\omega_0-2)}\frac{(N_q+1)}{2}$
	$\times\frac{\beta_+(E/\hbar\omega_0)}{\alpha(E/\hbar\omega_0-1)}\,p_F f_0$
$\vec{p}\rightarrow\vec{p}+\vec{q}\rightarrow\vec{p}$	$\frac{N_q}{2}\frac{\beta_+(E/\hbar\omega_0+1)}{\alpha(E/\hbar\omega_0)}\frac{(N_q+1)}{2}$
	$\times\frac{\beta_-(E/\hbar\omega_0)}{\alpha(E/\hbar\omega_0+1)}\,p_F f_0$
$\vec{p}\rightarrow\vec{p}-\vec{q}\rightarrow\vec{p}$	$\frac{(N_q+1)}{2}\frac{\beta_-(E/\hbar\omega_0-1)}{\alpha(E/\hbar\omega_0)}\frac{N_q}{2}$
	$\times\frac{\beta_+(E/\hbar\omega_0)}{\alpha(E/\hbar\omega_0-1)}\,p_F f_0$

where function $\beta_+(E/\hbar\omega_0)$ is self-defined by Eq. (32).

(iii) The four terms in $H\tau H\tau p_F f_0$ can now be deduced easily and they are shown in Table I.

(iv) The general term $H\tau H\tau\cdots H\tau$ n times operating on $p_F f_0$ can be obtained with the aid of the "mesh" diagram in Fig. 1. In this diagram the points represent the states of the electrons. We assign the correspondence between the transition, transition diagram, and mathematical expression shown in Table II.

To obtain, for example, $H\tau H\tau H\tau H\tau H\tau p_F f_0$ from the "mesh" diagram, we start from the points 1, 2, 3, 4, 5, and 6 which correspond, respectively, to the states $\vec{p}+5\vec{q}, \vec{p}+3\vec{q}, \vec{p}+\vec{q}, \vec{p}-\vec{q}, \vec{p}-3\vec{q}$, and $\vec{p}-5\vec{q}$, and find all possible paths by following the arrows through which each of these six points can reach the vertex 0. The value of a path is the product of the transitions traversed by the path. For example, the value of the path $2abcdo$ is

$$\frac{(N_q+1)}{2}\frac{\beta_+(E/\hbar\omega_0+4)}{\alpha(E/\hbar\omega_0+3)}$$

$$\times\frac{N_q}{2}\frac{\beta_-(E/\hbar\omega_0+3)}{\alpha(E/\hbar\omega_0+4)}\frac{N_q}{2}\frac{\beta_-(E/\hbar\omega_0+2)}{\alpha(E/\hbar\omega_0+3)}$$

$$\times\frac{N_q}{2}\frac{\beta_-(E/\hbar\omega_0+1)}{\alpha(E/\hbar\omega_0+2)}\frac{N_q}{2}\frac{\beta_-(E/\hbar\omega_0)}{\alpha(E/\hbar\omega_0+1)}\ .$$

The value of $H\tau H\tau H\tau H\tau H\tau\ p_F f_0$ is the sum of the values of all possible paths multiplied by $p_F f_0$, the number of all the paths being 2^5.

At a given temperature T most of the current is due to the drift of the electrons with energies close to $k_B T$. Thus an examination of the following general expressions for transitions,

$$\begin{aligned}A_1&=\frac{N_q}{2}\frac{\beta_-(T/\Theta+n-1)}{\alpha(T/\Theta+n)}\ ,\\A_2&=\frac{N_q}{2}\frac{\beta_-(T/\Theta-n-1)}{\alpha(T/\Theta-n)}\ ,\\A_3&=\frac{(N_q+1)}{2}\frac{\beta_+(T/\Theta+n+1)}{\alpha(T/\Theta+n)}\ ,\\A_4&=\frac{(N_q+1)}{2}\frac{\beta_+(T/\Theta-n+1)}{\alpha(T/\Theta-n)}\ ,\end{aligned}\tag{33}$$

which are functions of n, should give an indication of the rate of convergence of the series for f_1, where Θ is the Debye temperature of the polar-optical phonons. In Figs. 2 and 3 are shown two cases for $T=\Theta$ and $T=20\Theta$ with various degrees of the screening effect. A small value of E_s corresponds to a small screening effect. The value of E_s can be estimated as follows:

$$E_s\sim\frac{\hbar^2}{m\lambda_D^2}\sim\frac{\hbar^2Ne^2}{mk_BT}\ ,\tag{34}$$

where N is the electron concentration. For $m\simeq$

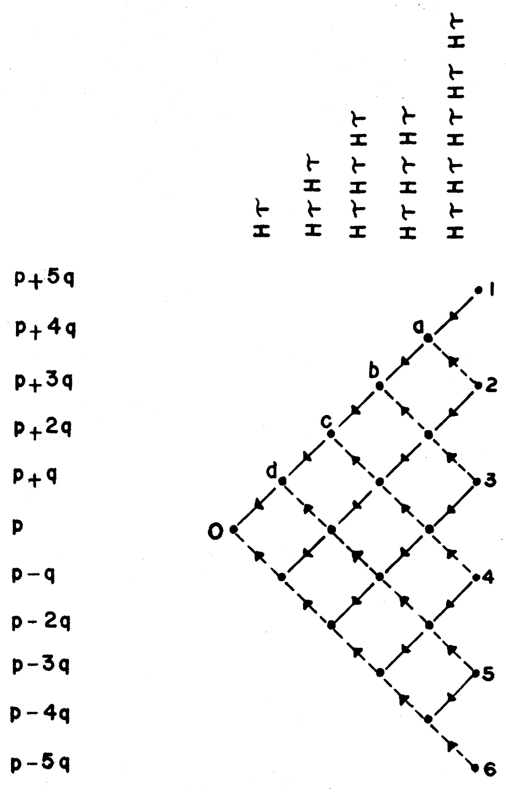
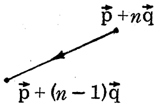
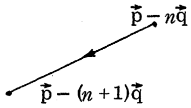
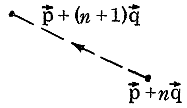
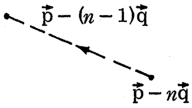


FIG. 1. Mesh diagram.

TABLE II. Correspondence between transition, transition diagram, and mathematical expression.

Transition	n	Transition diagram	Mathematical expression
$\vec{p}+n\vec{q}\to\vec{p}+(n-1)\vec{q}$	$\geqslant 1$		$\frac{N_q}{2}\,\frac{\beta_-(E/\hbar\omega_0+n-1)}{\alpha(E/\hbar\omega_0+n)}$
$\vec{p}-n\vec{q}\to\vec{p}-(n+1)\vec{q}$	$\geqslant 0$		$\frac{N_q}{2}\,\frac{\beta_-(E/\hbar\omega_0-n-1)}{\alpha(E/\hbar\omega_0-n)}$
$\vec{p}+n\vec{q}\to\vec{p}+(n+1)\vec{q}$	$\geqslant 0$		$\frac{(N_q+1)}{2}\,\frac{\beta_+(E/\hbar\omega_0+n+1)}{\alpha(E/\hbar\omega_0+n)}$
$\vec{p}-n\vec{q}\to\vec{p}-(n-1)\vec{q}$	$\geqslant 0$		$\frac{(N_q+1)}{2}\,\frac{\beta_+(E/\hbar\omega_0-n+1)}{\alpha(E/\hbar\omega_0-n)}$

free-electron mass, $N\simeq 10^{18}\,\mathrm{cm}^{-3}$, and at $T\simeq 293$ °K, $E_s\simeq 1$ eV. Thus for nondegenerate semiconductors E_s would be considerably smaller than 1 eV. With $E_s\simeq 10^{-8}$ eV, the polar-optical scattering is practically unscreened. It can be seen in Figs. 2 and 3 that the values of A_1, A_2, A_3, and A_4 increase with increasing screening effect. A reasonable estimate of the ratio $(H\tau)^m\,p_F f_0/(p_F f_0)$ can be obtained by 2^m (average value of A_1, A_2, A_3, and $A_4)^m$, the factor 2^m being the number of paths. Since 2× (average value of A_1, A_2, A_3, and A_4) is less than unity, the series for f_1 as given in Eq. (10) is a monotonic series. This is, in fact, expected in view of the physical interpretation given to the term $H\tau\,\partial f_0/\partial p_F$ in Eq. (12). Figure 2 shows that at $T=\Theta$ the truncation of the series at the term $H\tau H\tau\,\partial f_0/\partial p_F$ introduces only an error of about 5%, and Fig. 3 shows that at $T=20\Theta$, $E_s=10^{-2}$ eV, the truncation of the series at the term $H\tau H\tau\,\partial f_0/\partial p_F$ introduces an error greater than 10%, while truncating at the term $(H\tau)^5\,\partial f_0/\partial p_F$, the error reduces to about 6%. For $T=\Theta$, $E_s=10^{-2}$ eV, and electron energy $E=20\hbar\omega_0$, Fig. 4 shows that the truncation of the series at the term $(H\tau)^5\,\partial f_0/\partial p_F$ intro-

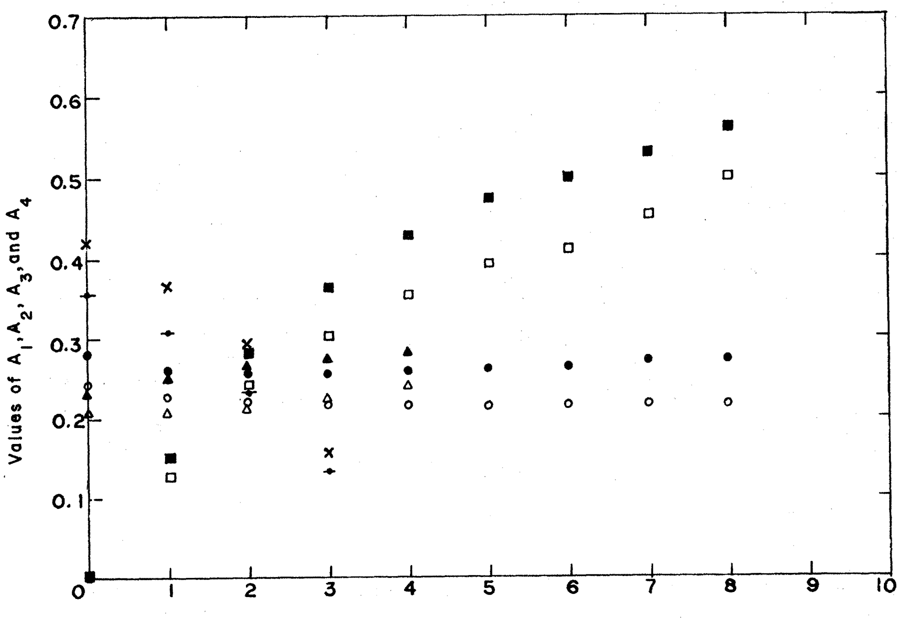


FIG. 2. Values of A_1 and A_2 as functions of n at $T=\Theta$ and $E=\hbar\omega_0$, and values of A_3 and A_4 as functions of n at $T=\Theta$ and $E=5\hbar\omega_0$. A_1, A_2, A_3, and A_4 are indicated, respectively, by ●, ▲, ■, and × for $E_s=10^{-2}$ eV; and by ○, △, □, and -•- for $E_s=10^{-8}$ eV.

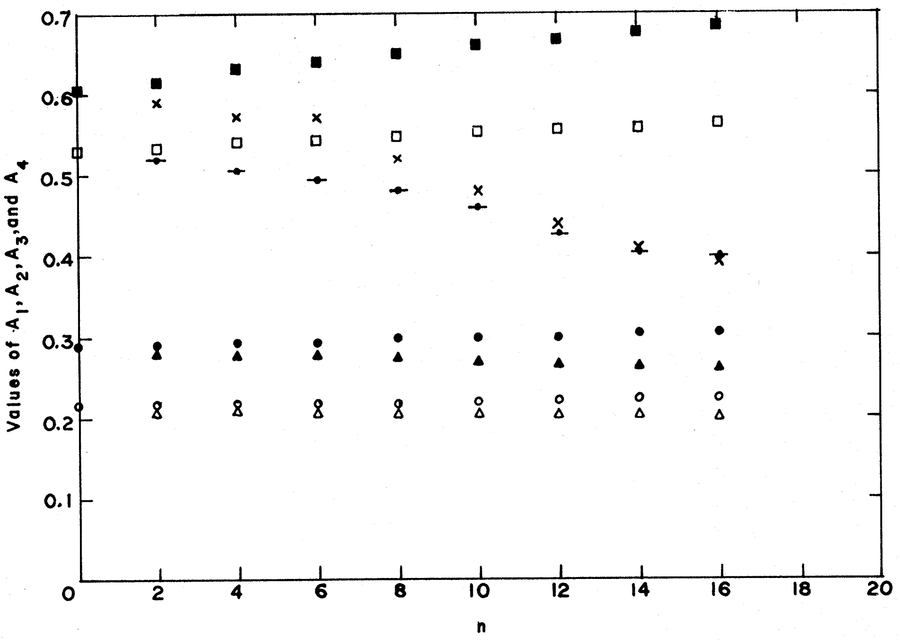


FIG. 3. Values of A_1, A_2, A_3, and A_4 as functions of n at $T=20\Theta$ and $E=20\hbar\omega_0$. A_1, A_2, A_3, and A_4 are indicated, respectively, by ●, ▲, ■, and × for $E_s=10^{-2}$ eV; and by ○, △, □, and -•- for $E_s=10^{-8}$ eV.

duces an error of about 10% and truncating at the term $(H\tau)^{10}\,\partial f_0/\partial p_F$ the error reduces to about 5%. From these results it can be concluded that the normal consideration for the polar scattering to be nearly elastic at $T\gg\Theta$ or at $T=\Theta$ but $E\gg\hbar\omega_0$ is incorrect because it fails to appreciate the effect of the extreme anisotropy.

C. Mixed Scattering

When there are several types of electron-phonon scatterings which are equally important in limiting the electron mobility, the operators τ and H have to be replaced, respectively, by the sums of operators $(\sum_i 1/\tau)^{-1}$ and $\sum_i H_i$, where the index i refers to the different types of electron-phonon scatterings. Furthermore, if the scatterings which are isotropic are indexed by r, and those nearly elastic and isotropic by e, then

$$\frac{1}{\tau}=\sum_i\frac{1}{\tau_{ir}}+\sum_j\frac{1}{\tau_{je}}\,, \tag{35}$$

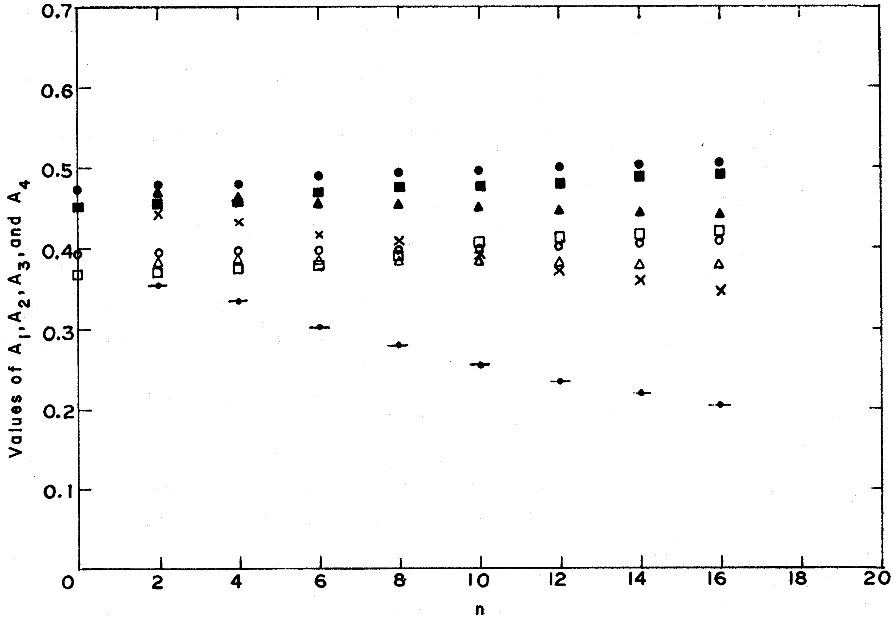


FIG. 4. Values of A_1, A_2, A_3, and A_4 as functions of n at $T=\Theta$ and $E=20\hbar\omega_0$. A_1, A_2, A_3, and A_4 are indicated, respectively, by ●, ▲, ■, and × for $E_s=10^{-2}$ eV; and by ○, △, □, and -•- for $E_s=10^{-8}$ eV.

$$H = \sum_i H_{ir} + \sum_j H_{je} \quad . \tag{36}$$

Substitution of Eqs. (35) and (36) into Eq. (24) gives

$$\sum_i \frac{1}{\tau_{ir} g(E)} = \sum_i \frac{1}{\tau_{ir}} + \sum_j \sum_{\vec{p}'} [W_{je}(\vec{p} \to \vec{p}')(1 - \cos\phi)]. \tag{37}$$

Equation (37) is equivalent to Matthiessen's rule which states that the total resistivity equals the sum of the component resistivities due to different scattering mechanisms acting independently.

If one of the scattering mechanisms is neither isotropic nor nearly elastic and isotropic, we would have to resort to Eq. (10) using the appropriate sums for τ and H. Though the general term $(H\tau)^n\, \partial f_0/\partial p_F$ for this case would become more complicated, the series for f_1 is expected to converge more rapidly than that for the case involving polar-optical scattering alone.

IV. SOLUTION OF LINEARIZED BOLTZMANN EQUATION IN PRESENCE OF ELECTRIC AND MAGNETIC FIELDS

The linearized Boltzmann equation in the presence of applied electric and magnetic fields is given by

$$e\left(\vec{F} + \frac{\vec{v} \times \vec{B}}{c}\right) \cdot \frac{\partial f}{\partial \vec{p}} + \frac{f}{\tau} - Hf = 0 \quad , \tag{38}$$

where $\vec{B}$ is the magnetic field, c is the speed of light, and $\vec{v}$ is the electron velocity which is equal to $\partial E/\partial \vec{p}$. For low electric fields Eq. (38) becomes

$$e\,\frac{\partial f_0}{\partial p_F} + e\,\frac{\vec{v} \times \vec{B}}{c} \cdot \frac{\partial f_1}{\partial \vec{p}} + \frac{f_1}{\tau} - Hf = 0. \tag{39}$$

Applying the same iterative technique as described in Sec. II, the exact solution of Eq. (39) is

$$f_1 = -e\tau\left[1 + \left(H - e\,\frac{\vec{v} \times \vec{B}}{c} \cdot \frac{\partial}{\partial \vec{p}}\right)\tau + \left(H - e\,\frac{\vec{v} \times \vec{B}}{c} \cdot \frac{\partial}{\partial \vec{p}}\right) \times \tau\left(H - e\,\frac{\vec{v} \times \vec{B}}{c} \cdot \frac{\partial}{\partial \vec{p}}\right)\tau + \cdots\right]\frac{\partial f_0}{\partial p_F} \quad . \tag{40}$$

The effect of the magnetic field is described by the terms having the common factor

$$-e\,\frac{\vec{v} \times \vec{B}}{c} \cdot \frac{\partial}{\partial \vec{p}}\left(\tau\,\frac{\partial f_0}{\partial p_F}\right)$$

which can be written as

$$-e\,\frac{\vec{v} \times \vec{B}}{c} \cdot \frac{\partial}{\partial \vec{p}}\left(\tau\,\frac{\partial f_0}{\partial p_F}\right) = e\,\frac{\vec{v} \times \vec{B}}{c} \cdot \left[-\frac{\partial \tau}{\partial \vec{p}}\,\frac{\partial f_0}{\partial p_F} + \frac{\partial}{\partial \vec{p}}\left(\frac{\partial E}{\partial p_F}\right)\frac{f_0}{k_B T}\,\tau\right] . \tag{41}$$

It should be noted that when $\vec{F}$ is parallel to $\vec{B}$, the term $\vec{v} \times \vec{B} \cdot \partial\tau/\partial\vec{p} = 0$ unless τ is dependent on the direction of $\vec{p}$ in a plane normal to $\vec{B}$, and the term $\vec{v} \times \vec{B} \cdot (\partial/\partial\vec{p})(\partial E/\partial p_F) = 0$ unless the effective mass is a tensorial quantity. These are well-known facts regarding the nonvanishing of longitudinal magnetoresistance. In the following we shall treat in detail the semiconductors in which m is scalar and τ is independent of the direction of $\vec{p}$. For this case it can be assumed without loss of generality that $\vec{B}$ is perpendicular to $\vec{F}$. By choosing $\vec{B}$ in the direction along z axis, we obtain

$$-e\,\frac{\vec{v} \times \vec{B}}{c} \cdot \frac{\partial}{\partial \vec{p}}\left(\tau\,\frac{\partial f_0}{\partial p_F}\right) = \frac{f_0}{k_B T}\,\tau\omega(-v_x + v_y) \quad , \tag{42}$$

and to the second order in B, we have

$$-e\,\frac{\vec{v} \times \vec{B}}{c} \cdot \frac{\partial}{\partial \vec{p}}\left(\frac{f_0}{k_B T}\,\tau^2\omega(-v_x + v_y)\right) = \left(\frac{f_0\tau}{k_B T}\right)^2 \omega(-v_x - v_y) \ , \tag{43}$$

where v_x and v_y are the electron velocities in x and y direction, respectively, and $\omega = eB/mc$, the cyclotron resonance angular frequency. Equations (42) and (43) can now be generalized to any order in B as follows:

$$\left(-e\,\frac{\vec{v} \times \vec{B}}{c} \cdot \frac{\partial}{\partial \vec{p}}\,\tau\right)^{2n+1}\left(\frac{\partial f_0}{\partial p_F}\right) = \left(\frac{f_0\tau}{k_B T}\right)^{2n+1} \times \omega^{2n+1}(-v_x + v_y)(-1)^n \text{ for } n = 0,\ 1,\ 2,\ldots, \tag{44}$$

$$\left(-e\,\frac{\vec{v} \times \vec{B}}{c} \cdot \frac{\partial}{\partial \vec{p}}\,\tau\right)^{2n}\left(\frac{\partial f_0}{\partial p_F}\right) = \left(\frac{f_0\tau}{k_B T}\right)^{2n} \times \omega^{2n}(v_x + v_y)(-1)^n \text{ for } n = 1,\ 2,\ 3,\ \ldots \ . \tag{45}$$

We can now determine the distribution function f_1 for various types of scatterings:

(*i*) *Isotropic scattering.* The solution for f_1 is

$$f_1 = -e\tau\left[1 + \left(-e\,\frac{\vec{v} \times \vec{B}}{c} \cdot \frac{\partial}{\partial \vec{p}}\,\tau\right) + \left(-e\,\frac{\vec{v} \times \vec{B}}{c} \cdot \frac{\partial}{\partial \vec{p}}\,\tau\right)^2 + \cdots\right]\frac{\partial f_0}{\partial p_F} \ . \tag{46}$$

The solution to any order in B can be obtained using Eqs. (44) and (45).

(*ii*) *Nearly elastic and isotropic scattering.* If it is assumed that the scattering takes place on constant energy surfaces, the operators $H\tau$ and

$$-e\,\frac{\vec{v} \times \vec{B}}{c} \cdot \frac{\partial}{\partial \vec{p}}\,\tau$$

commute. Then to the first order in B, f_1 is given by

$$f_1 = -e\tau\left[(1+H\tau+H\tau H\tau+\cdots)+\left(-e\,\frac{\vec{\mathrm{v}}\times\vec{\mathrm{B}}}{c}\cdot\frac{\partial}{\partial\vec{\mathrm{p}}}\,\tau\right)\times[1+2H\tau+3H\tau H\tau+\cdots+(n+1)(H\tau)^n+\cdots]\right]\frac{\partial f_0}{\partial p_F}\,. \tag{47}$$

By writing

$$[1+2H\tau+3H\tau+\cdots+(n+1)(H\tau)^n+\cdots]\,\frac{\partial f_0}{\partial p_F} = -G(E)\,\frac{f_0}{k_BT}\,v_F\,, \tag{48}$$

where v_F is the velocity of the electron in the direction of F, $G(E)$ can be determined by the following equation:

$$-\frac{\partial f_0}{\partial p_F} = (1-H\tau)^2\left(G(E)\,\frac{f_0}{k_BT}\,v_F\right) \tag{49}$$

which gives

$$G(E) = 1/[1-\tau\textstyle\sum_{\vec{\mathrm{p}}'}W(\vec{\mathrm{p}}\rightarrow\vec{\mathrm{p}}')\cos\phi]^2\,. \tag{50}$$

Thus f_1 to the first order in B is

$$f_1 = \left(e\Big/\sum_{\vec{\mathrm{p}}'}[W(\vec{\mathrm{p}}\rightarrow\vec{\mathrm{p}}')(1-\cos\phi)]\right)\frac{f_0}{k_BT}\,v_F - \left[e\Big/\left(\sum_{\vec{\mathrm{p}}'}[W(\vec{\mathrm{p}}\rightarrow\vec{\mathrm{p}}')(1-\cos\phi)]\right)^2\right]\times\frac{f_0}{k_BT}\,\omega(-v_x+v_y)\,. \tag{51}$$

To calculate f_1 to the nth order in B we proceed as follows. The nth order terms in B can be obtained from the expansion of the following terms:

$$\left(H\tau - e\,\frac{\vec{\mathrm{v}}\times\vec{\mathrm{B}}}{c}\cdot\frac{\partial}{\partial\vec{\mathrm{p}}}\,\tau\right)^b\frac{\partial f_0}{\partial p_F}\,,\qquad b\geqslant n\,.$$

By expanding binomially

$$\left(H\tau - e\,\frac{\vec{\mathrm{v}}\times\vec{\mathrm{B}}}{c}\cdot\frac{\partial}{\partial\vec{\mathrm{p}}}\,\tau\right)^b\,,$$

the general term in the nth order of B is

$$\left(-e\,\frac{\vec{\mathrm{v}}\times\vec{\mathrm{B}}}{c}\cdot\frac{\partial}{\partial\vec{\mathrm{p}}}\,\tau\right)^n C_n^{n+m}\,(H\tau)^m\,\frac{\partial f_0}{\partial p_F}\,,$$

where C_n^{n+m} is the binomial coefficient and $m=0, 1, 2, \ldots$. By summing up all these terms and recognizing that

$$1+C_n^{n+1}\,H\tau+\cdots+C_n^{n+m}\,(H\tau)^m+\cdots = \frac{1}{(1-H\tau)^{n+1}}\,,$$

the total contribution to f_1 in the nth order of B is

$$f_1(B^n) = -e\tau\left(-e\,\frac{\vec{\mathrm{v}}\times\vec{\mathrm{B}}}{c}\cdot\frac{\partial}{\partial\vec{\mathrm{p}}}\,\tau\right)^n\times\left[1\Big/\left(1-\tau\sum_{\vec{\mathrm{p}}'}[W(\vec{\mathrm{p}}\rightarrow\vec{\mathrm{p}}')\cos\phi]\right)^{n+1}\right]\frac{\partial f_0}{\partial p_F}\,. \tag{52}$$

In particular, the term $f_1(B^2)$ in the second order of B is of importance in calculating the magnetoresistance. This term is

$$f_1(B^2) = -\left[e\Big/\left(\sum_{\vec{\mathrm{p}}'}[W(\vec{\mathrm{p}}\rightarrow\vec{\mathrm{p}}')(1-\cos\phi)]\right)^3\right]\times\frac{f_0}{k_BT}\,\omega^2(v_x+v_y)\,. \tag{53}$$

Equations (46), (51), and (53) are identical to those obtained from the relaxation-time approximation.

(*iii*) *Polar-optical scattering*. For this case we first consider the terms which are of first order in B and contain the operator $(H\tau)^m$ as follows:

$$\left(-e\,\frac{\vec{\mathrm{v}}\times\vec{\mathrm{B}}}{c}\cdot\frac{\partial}{\partial\vec{\mathrm{p}}}\,\tau\right)(H\tau)^m\,\frac{\partial f_0}{\partial p_F}\,,$$

$$H\tau\left(-e\,\frac{\vec{\mathrm{v}}\times\vec{\mathrm{B}}}{c}\cdot\frac{\partial}{\partial\vec{\mathrm{p}}}\,\tau\right)(H\tau)^{m-1}\,\frac{\partial f_0}{\partial p_F}\,,\;\ldots$$

$$(H\tau)^b\left(-e\,\frac{\vec{\mathrm{v}}\times\vec{\mathrm{B}}}{c}\cdot\frac{\partial}{\partial\vec{\mathrm{p}}}\,\tau\right)(H\tau)^{m-b}\,\frac{\partial f_0}{\partial p_F}\,,\;\ldots$$

$$(H\tau)^m\left(-e\,\frac{\vec{\mathrm{v}}\times\vec{\mathrm{B}}}{c}\cdot\frac{\partial}{\partial\vec{\mathrm{p}}}\,\tau\right)\frac{\partial f_0}{\partial p_F}\,.$$

The term $(H\tau)^m\,\partial f_0/\partial p_F$ can be evaluated by computing the values of all the paths in the "mesh" diagram described in Sec. III B. If $\tau(x_{i,j})$ represents τ evaluated at the state represented by point $x_{i,j}$, where j indexes the points on the path i in the "mesh" diagram, then the sum of all the terms listed above can be expressed as

$$\sum_{\substack{\text{All paths } i\\ \text{in } (H\tau)^m\,\partial f_0/\partial p_F}}\left[(\text{value of path } i)\left(\sum_{\substack{\text{All points } j\\ \text{on path } i}}\tau(x_{i,j})\right)\right]\frac{f_0}{k_BT}\,\omega(-v_x+v_y)\,. \tag{54}$$

As an illustration we have

$$\left[H\tau\left(-e\,\frac{\vec{\mathrm{v}}\times\vec{\mathrm{B}}}{c}\cdot\frac{\partial}{\partial\vec{\mathrm{p}}}\,\tau\right)+\left(-e\,\frac{\vec{\mathrm{v}}\times\vec{\mathrm{B}}}{c}\cdot\frac{\partial}{\partial\vec{\mathrm{p}}}\,\tau\right)H\tau\right]\frac{\partial f_0}{\partial p_F}$$

$$=\left(\frac{N_q}{2}\,\frac{\beta_-(E/\hbar\omega_0)}{\alpha(E/\hbar\omega_0+1)}\,[\tau(E+\hbar\omega_0)+\tau(E)]+\frac{(N_q+1)}{2}\,\frac{\beta_+(E/\hbar\omega_0)}{\alpha(E/\hbar\omega_0-1)}\,[\tau(E-\hbar\omega_0)+\tau(E)]\right)\frac{f_0}{k_BT}\,\omega(-v_x+v_y)\;, \tag{55}$$

$$\left[H\tau H\tau\left(-e\,\frac{\vec{\mathrm{v}}\times\vec{\mathrm{B}}}{c}\cdot\frac{\partial}{\partial\vec{\mathrm{p}}}\,\tau\right)+H\tau\left(-e\,\frac{\vec{\mathrm{v}}\times\vec{\mathrm{B}}}{c}\cdot\frac{\partial}{\partial\vec{\mathrm{p}}}\,\tau\right)H\tau+\left(-e\,\frac{\vec{\mathrm{v}}\times\vec{\mathrm{B}}}{c}\cdot\frac{\partial}{\partial\vec{\mathrm{p}}}\,\tau\right)H\tau H\tau\right]\frac{\partial f_0}{\partial p_F}$$

$$\begin{aligned}=\Bigg(&\frac{N_q}{2}\,\frac{\beta_-(E/\hbar\omega_0+1)}{\alpha(E/\hbar\omega_0+2)}\cdot\frac{N_q}{2}\,\frac{\beta_-(E/\hbar\omega_0)}{\alpha(E/\hbar\omega_0+1)}\,[\tau(E+2\hbar\omega_0)+\tau(E+\hbar\omega_0)+\tau(E)]+\frac{(N_q+1)}{2}\,\frac{\beta_+(E/\hbar\omega_0+1)}{\alpha(E/\hbar\omega_0)}\,\frac{N_q}{2}\\&\times\frac{\beta_-(E/\hbar\omega_0)}{\alpha(E/\hbar\omega_0+1)}\,[\tau(E)+\tau(E+\hbar\omega_0)+\tau(E)]+\frac{N_q}{2}\,\frac{\beta_-(E/\hbar\omega_0-1)}{\alpha(E/\hbar\omega_0)}\,\frac{(N_q+1)}{2}\,\frac{\beta_+(E/\hbar\omega_0)}{\alpha(E/\hbar\omega_0-1)}\,[\tau(E)+\tau(E-\hbar\omega_0)+\tau(E)]\\&+\frac{(N_q+1)}{2}\,\frac{\beta_+(E/\hbar\omega_0-1)}{\alpha(E/\hbar\omega_0-2)}\,\frac{(N_q+1)}{2}\,\frac{\beta_+(E/\hbar\omega_0)}{\alpha(E/\hbar\omega_0-1)}\,[\tau(E-2\hbar\omega_0)+\tau(E-\hbar\omega_0)+\tau(E)]\Bigg)\frac{f_0}{k_BT}\,\omega(-v_x-v_y)\;.\end{aligned}\tag{56}$$

Now we can write a general formula to calculate the sum of the terms which are of either even or odd order in B and contain the operator $(H\tau)^m$ as

$$\sum_{\substack{\text{all paths } i\\ \text{in } (H\tau)^m\,\partial f_0/\partial p_F}}(\text{value of path } i)\begin{cases}\times\left(\displaystyle\sum_{\substack{\text{all points } j\\ \text{on path } i}}\tau(x_{i,j})\right)^{2n}\left(\dfrac{f_0}{k_BT}\right)^{2n}\omega^{2n}(v_x+v_y)(-1)^n & \text{for the case in even order in } B\\[2ex] \times\left(\displaystyle\sum_{\substack{\text{all points } j\\ \text{on path } i}}\tau(x_{i,j})\right)^{2n+1}\left(\dfrac{f_0}{k_BT}\right)^{2n+1}\omega^{2n+1}(-v_x+v_y)(-1)^n & \text{for the case in odd order in } B.\end{cases}\tag{57}$$

It should be noted that for mixed scattering the same principle as described in Sec. IIIC can be applied to calculate the over-all distribution function in the presence of both electric and magnetic fields.

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